

THE BEST LOWER AND UPPER BOUNDS OF HARMONIC SEQUENCE

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ABSTRACT. For any natural number $n \in \mathbb{N}$,

$$\frac{1}{2n + \frac{1}{1-\gamma} - 2} \leq \sum_{i=1}^n \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}, \quad (1)$$

where $\gamma = 0.57721566490153286 \dots$ denotes Euler's constant. The constants $\frac{1}{1-\gamma} - 2$ and $\frac{1}{3}$ are the best possible.

1. INTRODUCTION

Let n be a natural number, then we have

$$\frac{1}{2n} - \frac{1}{8n^2} < \sum_{i=1}^n \frac{1}{i} - \ln n - \gamma < \frac{1}{2n}, \quad (2)$$

where $\gamma = 0.57721566 \dots$ is Euler's constant.

The inequality (2) is called in literature Franel's inequality [4, Ex. 18]. Because of the well known importance of the harmonic sequence $\sum_{i=1}^n \frac{1}{i}$, there exists a very rich literature on inequalities of the harmonic sequence $\sum_{i=1}^n \frac{1}{i}$. For example, [1], [3, pp. 68–78] and references therein.

L. Tóth and S. Mare in [5, p. 264] proposed the following problems:

- (1) Prove that for every positive integer n we have

$$\frac{1}{2n + \frac{2}{5}} < 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}, \quad (3)$$

where γ is Euler's constant.

- (2) Show that $\frac{2}{5}$ can be replaced by a slightly smaller number, but that $\frac{1}{3}$ cannot be replaced by a slightly larger number.

In 1997 and 1999, K. Wu and B.-Ch. Yang in [8] and Sh.-R. Wei and B.-Ch. Yang in [7] verified inequality (3).

In this short note, we shall give the best lower and upper bounds of the sequence $\sum_{i=1}^n \frac{1}{i} - \ln n - \gamma$ and refine inequality (3), obtain the following

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Theorem 1. For any natural number $n \in \mathbb{N}$, we have

$$\frac{1}{2n + \frac{1}{1-\gamma} - 2} \leq \sum_{i=1}^n \frac{1}{i} - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}, \quad (4)$$

where $\gamma = 0.57721566490153286 \dots$ denotes Euler's constant. The constants $\frac{1}{1-\gamma} - 2$ and $\frac{1}{3}$ are the best possible.

2. LEMMA

In order to prove inequality (3), the following lemma is necessary.

Lemma 1. For $x > 0$, we have

$$\frac{1}{2x} - \frac{1}{12x^2} < \psi(x+1) - \ln x < \frac{1}{2x} \quad (5)$$

and

$$\frac{1}{2x^2} - \frac{1}{6x^3} < \frac{1}{x} - \psi'(x+1) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5}, \quad (6)$$

where $\psi = \frac{\Gamma'}{\Gamma}$ is the logarithmic derivative of the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \quad (7)$$

Proof. It is a well known fact ([1] and [6, p. 103]) that for $x > 0$ and a nonnegative integer m ,

$$\psi(x+1) = \psi(x) + \frac{1}{x} \quad (8)$$

and

$$\frac{m!}{x^{m+1}} = \int_0^\infty t^m e^{-xt} dt. \quad (9)$$

The first Binet's formula ([1] and [6, p. 106]) states that for $x > 0$

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} - \int_0^\infty \left(\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}}\right) \frac{e^{-xt}}{t} dt. \quad (10)$$

Differentiating (10), integrating by part and using formulas (9) and (8), it is deduced that

$$\psi(x+1) - \ln x = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-xt} dt. \quad (11)$$

Using formulas (9) and (11) and the series expansion of e^x at $x = 0$ yields

$$\begin{aligned} & \psi(x+1) - \ln x - \frac{1}{2x} + \frac{1}{12x^2} \\ &= \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} + \frac{1}{12}t\right) e^{-xt} dt \\ &= \int_0^\infty \frac{12(e^t - 1) - 12t - 6t(e^t - 1) + t^2(e^t - 1)}{12t(e^t - 1)} e^{-xt} dt \\ &= \int_0^\infty \left[\frac{1}{12t(e^t - 1)} \sum_{n=5}^\infty \frac{(n-3)(n-4)}{n!} t^n \right] e^{-xt} dt \\ &> 0 \end{aligned} \quad (12)$$

and

$$\begin{aligned}\psi(x+1) - \ln x - \frac{1}{2x} &= \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} \right) e^{-xt} dt \\ &= - \int_0^\infty \left[\frac{1}{2t(e^t - 1)} \sum_{n=3}^\infty \frac{n-2}{n!} t^n \right] e^{-xt} dt \\ &< 0.\end{aligned}\quad (13)$$

Hence, inequality (5) follows.

Differentiation of (11) immediately produces

$$\frac{1}{x} - \psi'(x+1) = \int_0^\infty \left(1 - \frac{t}{e^t - 1} \right) e^{-xt} dt. \quad (14)$$

Exploiting formulas (9) and (14) and the series expansion of e^x at $x = 0$ yields

$$\begin{aligned}&\frac{1}{x} - \psi'(x+1) - \frac{1}{2x^2} + \frac{1}{6x^3} \\ &= \int_0^\infty \left(1 - \frac{t}{e^t - 1} - \frac{1}{2}t + \frac{1}{12}t^2 \right) e^{-xt} dt \\ &= \int_0^\infty \left[\frac{1}{12(e^t - 1)} \sum_{n=5}^\infty \frac{(n-3)(n-4)}{n!} t^n \right] e^{-xt} dt \\ &> 0\end{aligned}\quad (15)$$

and

$$\begin{aligned}&\frac{1}{x} - \psi'(x+1) - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} \\ &= \int_0^\infty \left(1 - \frac{t}{e^t - 1} - \frac{1}{2}t + \frac{1}{12}t^2 - \frac{1}{720}t^4 \right) e^{-xt} dt \\ &= \int_0^\infty \left[\frac{1}{720(e^t - 1)} \sum_{n=7}^\infty \left(\frac{720}{n!} - \frac{360}{(n-1)!} + \frac{60}{(n-2)!} - \frac{1}{(n-4)!} \right) t^n \right] e^{-xt} dt.\end{aligned}\quad (16)$$

Noticing that for $n \geq 7$,

$$\begin{aligned}&\frac{720}{n!} - \frac{360}{(n-1)!} + \frac{60}{(n-2)!} - \frac{1}{(n-4)!} \\ &= - \frac{120 + 218(n-7) + 119(n-7)^2 + 22(n-7)^3 + (n-7)^4}{n!} < 0,\end{aligned}\quad (17)$$

we obtain

$$\frac{1}{x} - \psi'(x+1) - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < 0. \quad (18)$$

Therefore, inequality (6) holds. The proof is complete. \square

3. PROOF OF THEOREM 1

In [1], [2, p. 593] and [6, p. 104] it is given that $\psi(n) = \sum_{k=1}^{n-1} \frac{1}{k} - \gamma$. Thus, inequality (4) can be rearranged as

$$\frac{1}{3} < \frac{1}{\psi(n+1) - \ln n} - 2n \leq \frac{1}{1 - \gamma} - 2. \quad (19)$$

Define for $x > 0$

$$\phi(x) = \frac{1}{\psi(x+1) - \ln x} - 2x. \quad (20)$$

Differentiating ϕ and utilizing (5) and (6) reveals that for $x > \frac{12}{5}$,

$$\begin{aligned}
& (\psi(x+1) - \ln x)^2 \phi'(x) \\
&= \frac{1}{x} - \psi'(x+1) - 2(\psi(x+1) - \ln x)^2 \\
&< \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - 2\left(\frac{1}{2x} - \frac{1}{12x^2}\right)^2 \\
&= \frac{12-5x}{360x^5} < 0,
\end{aligned} \tag{21}$$

and $\phi(x)$ decreases with $x > \frac{12}{5}$.

Straightforward calculation produces

$$\phi(1) = \frac{1}{1-\gamma} - 2 = 0.36527211862544155 \dots, \tag{22}$$

$$\phi(2) = \frac{1}{\frac{3}{2} - \gamma - \ln 2} - 4 = 0.35469600731465752 \dots, \tag{23}$$

$$\phi(3) = \frac{1}{\frac{11}{6} - \gamma - \ln 3} - 6 = 0.34898948531361115 \dots. \tag{24}$$

Therefore, the sequence

$$\phi(n) = \frac{1}{\psi(n+1) - \ln n} - 2n, \quad n \in \mathbb{N} \tag{25}$$

is decreasing strictly, and for $n \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} \phi(n) < \phi(n) \leq \phi(1) = \frac{1}{1-\gamma} - 2. \tag{26}$$

Making use of approximating expansion of ψ in [1], [2, p. 594], or [6, p. 108] gives

$$\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4}) \quad (x \rightarrow \infty), \tag{27}$$

and then

$$\lim_{n \rightarrow \infty} \phi(n) = \lim_{x \rightarrow \infty} \phi(x) = \lim_{x \rightarrow \infty} \frac{\frac{1}{3} + O(x^{-2})}{1 + O(x^{-1})} = \frac{1}{3}. \tag{28}$$

The proof is complete.

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